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# Exact Solutions of Klein-Gordon Equation with Exponential Scalar and Vector Potentials

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**Abstract** We obtain the exact analytical solution of the Klein-Gordon equation for the exponential vector and scalar potentials by using the asymptotic iteration method. For the scalar potential greater than the vector potential case, the exact bound state energy eigenvalues and corresponding eigenfunctions are presented. The bound state eigenfunction solutions are obtained in terms of the confluent hypergeometric functions.

**Keywords** Asymptotic iteration method · Klein-Gordon equation · Equal vector and scalar potentials · Eigenvalues and eigenfunctions

## 1 Introduction

Since the exact solutions of the Klein-Gordon and Dirac equations with any potentials play an important role in relativistic quantum physics, there has been an increased interest in finding exact solutions of these equations with physical potentials by using different technics [4–16], i.e., supersymmetry (SUSY) [1], supersymmetric WKB approach [2], Nikiforov-Uvarov methods [3]. In the literature, there have been many studies in obtaining the exact solutions of the Klein-Gordon and Dirac equations with equally mixed potentials for some potentials such as the three-dimensional harmonic oscillator [4], Hulthén potential [5], pseudo-harmonic oscillator [6], ring-shaped Kratzer-type potential [7], ring-shaped non-spherical oscillator [8], double ring-shaped oscillator [9], Hartmann potential [10], Rosen-Morse-type potential [11], generalized symmetrical double-well potential [12], Scarf-type potential [13], generalized Pöschl-Teller potential [14], etc.

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In the present article, we investigate the calculation of the bound state energy eigenvalues and the corresponding eigenfunctions of exactly solvable exponential scalar and vector potentials for the s-wave in the radial Klein-Gordon equation within the framework of the asymptotic iteration method (AIM). In the next section, we briefly outline AIM. In Sect. 3, the general bound state energy eigenvalues and the corresponding eigenfunction expressions for exponential scalar and vector potentials are given. Finally, in Sect. 4, we remark on these results.

## 2 The Asymptotic Iteration Method

AIM [15–20] is proposed and applied to solve the second-order differential equations of the form

$$y_n''(x) = \lambda_0(x) y_n'(x) + s_0(x) y_n(x), \tag{1}$$

where $\lambda_0(x) \neq 0$ and the prime denotes the derivative with respect to $x$. The variables, $s_0(x)$ and $\lambda_0(x)$, are sufficiently differentiable. To find a general solution to this equation, we differentiate (1) with respect to $x$ and find

$$y_n'''(x) = \lambda_1(x) y_n'(x) + s_1(x) y_n(x), \tag{2}$$

where

$$\begin{aligned} \lambda_1(x) &= \lambda_0'(x) + s_0(x) + \lambda_0^2(x), \\ s_1(x) &= s_0'(x) + s_0(x)\lambda_0(x). \end{aligned} \tag{3}$$

Similarly, the second derivative of (1) yields

$$y_n^{(4)}(x) = \lambda_2(x) y_n'(x) + s_2(x) y_n(x), \tag{4}$$

where

$$\begin{aligned} \lambda_2(x) &= \lambda_1'(x) + s_1(x) + \lambda_0(x)\lambda_1(x), \\ s_2(x) &= s_1'(x) + s_0(x)\lambda_1(x). \end{aligned} \tag{5}$$

Equation (1) can be easily iterated up to $(k+1)$th and $(k+2)$th derivatives, $k = 1, 2, 3, \ldots$. Therefore, we have

$$\begin{aligned} y_n^{(k+1)}(x) &= \lambda_{k-1}(x) y_n'(x) + s_{k-1}(x) y_n(x), \\ y_n^{(k+2)}(x) &= \lambda_k(x) y_n'(x) + s_k(x) y_n(x), \end{aligned} \tag{6}$$

where

$$\begin{aligned} \lambda_k(x) &= \lambda_{k-1}'(x) + s_{k-1}(x) + \lambda_0(x)\lambda_{k-1}(x), \\ s_k(x) &= s_{k-1}'(x) + s_0(x)\lambda_{k-1}(x), \end{aligned} \tag{7}$$

which are called as the recurrence relation. From the ratio of the $(k+2)$th and $(k+1)$th derivatives, we have

$$\frac{d}{dx}\ln[y_n^{(k+1)}(x)] = \frac{y_n^{(k+2)}(x)}{y_n^{(k+1)}(x)} = \frac{\lambda_k(x)[y_n'(x) + \frac{s_k(x)}{\lambda_k(x)} y_n(x)]}{\lambda_{k-1}(x)[y_n'(x) + \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} y_n(x)]}. \tag{8}$$

For sufficiently large $k$, if

$$\frac{s_k(x)}{\lambda_k(x)} = \frac{s_{k-1}(x)}{\lambda_{k-1}(x)} = \alpha(x), \tag{9}$$

which is the "asymptotic" aspect of the method, then, (8) is reduced to

$$\frac{d}{dx}\ln[y_n^{(k+1)}(x)] = \frac{\lambda_k(x)}{\lambda_{k-1}(x)}, \tag{10}$$

which yields

$$y_n^{(k+1)}(x) = C_1 \exp\left(\int \frac{\lambda_k(x)}{\lambda_{k-1}(x)}dx\right) = C_1\lambda_{k-1}(x)\exp\left(\int [\alpha(x)+\lambda_0(x)]dx\right), \tag{11}$$

where $C_1$ is the integration constant and the right hand side of (11) is obtained by using (9) and (10). By inserting (11) into (6), the first-order differential equation is obtained as

$$y_n'(x) + \alpha(x)y_n(x) = C_1 \exp\left(\int [\alpha(x)+\lambda_0(x)]dx\right). \tag{12}$$

This first order differential equation can easily be solved and the general solution of (1) can be obtained as:

$$y_n(x) = \exp\left(-\int^x \alpha(x_1)dx_1\right)\left[C_2 + C_1\int^x \exp\left(\int^{x_1}[\lambda_0(x_2)+2\alpha(x_2)]dx_2\right)dx_1\right]. \tag{13}$$

For a given potential, the radial Schrödinger equation is converted to the form of (1). Then, $s_0(x)$ and $\lambda_0(x)$ are determined and $s_k(x)$ and $\lambda_k(x)$ parameters are calculated by the recurrence relations given by (7).

The termination condition of the method in (9) can be arranged as

$$\Delta_k(x) = \lambda_k(x)s_{k-1}(x) - \lambda_{k-1}(x)s_k(x) = 0, \tag{14}$$

where $k$ shows the iteration number. For the exactly solvable potentials, the energy eigenvalues are obtained from the roots of (14) and the radial quantum number $n$ is equal to the iteration number $k$ for this case. For nontrivial potentials that have no exact solutions, for a specific $n$ principal quantum number, we choose a suitable $x_0$ point, determined generally as the maximum value of the asymptotic wave function or the minimum value of the potential [16, 17, 20, 21] and the approximate energy eigenvalues are obtained from the roots of (14) for sufficiently great values of $k$ with iteration for which $k$ is always greater than $n$ in these numerical solutions.

The general solution of (1) is given by (13). The first part of (13) gives us the polynomial solutions that are convergent and physical, whereas the second part of (13) gives us non-physical solutions that are divergent. Although (13) is the general solution of (1), we take the coefficient of the second part ($C_1$) as zero, in order to find the square integrable solutions. Therefore, the corresponding eigenfunctions can be derived from the following wave function generator for exactly solvable potentials:

$$y_n(x) = C_2 \exp\left(-\int^x \frac{s_n(x_1)}{\lambda_n(x_1)}dx_1\right), \tag{15}$$

where $n$ represents the principal quantum number.

### 3 Bound State Solutions of the Mixed Exponential Potential

In the relativistic atomic units ($\hbar = c = 1$), for a spinless particle, time independent radial s-wave Klein-Gordon equation [11, 22, 23] with the radial vector $V(r)$ and scalar $S(r)$ potentials is written as follows

$$\frac{d^2}{dr^2}u(r) + [(E - V(r))^2 - (m + S(r))^2]u(r) = 0, \tag{16}$$

where the radial wave function is $\psi(r) = \frac{u(r)}{r}$. If we take the exponential-type scalar and vector potentials case, the potentials can be given as

$$S(r) = -S_0 e^{-\gamma r}, \qquad V(r) = -V_0 e^{-\gamma r}, \tag{17}$$

where $S_0$, $V_0$ and $\gamma$ are constants. For the real bound state, we should choose $S_0 \geq V_0$ and $m > E$ [22, 24]. Substituting the potentials in (16), we get

$$\frac{d^2}{dr^2}u(r) - [K_1 e^{-2\gamma r} + K_2 e^{-\gamma r} - E^2 + m^2]u(r) = 0, \tag{18}$$

where

$$K_1 = S_0^2 - V_0^2, \qquad K_2 = -(2mS_0 + 2EV_0). \tag{19}$$

Let us introduce

$$V_K(r) = K_1 e^{-2\gamma r} + K_2 e^{-\gamma r}, \tag{20}$$

which is called Klein-Gordon-Morse potential [26]. If we take $V_0 = 0$, $V_K(r)$ turns into the standard Morse potential. Equation (18) takes the form

$$\left[-\frac{d^2}{dr^2} + V_K(r) - E^2 + m^2\right]u(r) = 0. \tag{21}$$

Defining a new variable such as $s = e^{-\gamma r}$, we can find

$$\left[\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds} - \chi^2 - \frac{\xi}{s} + \frac{\varepsilon_n}{s^2}\right]u(s) = 0, \tag{22}$$

where

$$\chi = -\frac{\sqrt{K_1}}{\gamma}, \qquad \xi = \frac{K_2}{\gamma^2}, \qquad \varepsilon_n = \frac{\sqrt{m^2 - E_n^2}}{\gamma}. \tag{23}$$

To solve (22) by using AIM, similar to references [11, 25], we propose the following physical wave function

$$u(s) = s^{\varepsilon_n} e^{\chi s} f(s). \tag{24}$$

Inserting this wave function, (22) gives us

$$\frac{d^2}{ds^2}f(s) = -\frac{(2\varepsilon_n + 2\chi s + 1)}{s}\frac{d}{ds}f(s) - \frac{(2\varepsilon_n \chi + \chi - \xi)}{s}f(s). \tag{25}$$

This differential equation has the same form as (1). Therefore, we can use AIM to get general solution of (25). Comparing (22) with (25), we obtain $\lambda_0(s)$ and $s_0(s)$ and by using the

recursion relation, we calculate $\lambda_k(s)$ and $s_k(s)$. Combining these results with the condition given by (14) yields

$$\begin{aligned} s_0\lambda_1 - s_1\lambda_0 = 0 \quad &\Rightarrow \quad \varepsilon_0 = \frac{\xi - \chi}{2\chi}, \\ s_1\lambda_2 - s_2\lambda_1 = 0 \quad &\Rightarrow \quad \varepsilon_1 = \frac{\xi - 3\chi}{2\chi}, \\ s_2\lambda_3 - s_3\lambda_2 = 0 \quad &\Rightarrow \quad \varepsilon_2 = \frac{\xi - 5\chi}{2\chi}, \quad \text{etc.} \end{aligned} \tag{26}$$

If we generalize the above equations, the eigenvalues are obtained as

$$\varepsilon_n = \frac{\xi - (2n+1)\chi}{2\chi}, \quad n = 0, 1, 2, \ldots. \tag{27}$$

Using (23), we can obtain the energy eigenvalues of the exponential-type scalar and vector potentials as

$$E_n^2 = m^2 - \left(\gamma n + \frac{\gamma}{2} + \frac{1}{2}\frac{K_2}{\sqrt{K_1}}\right)^2. \tag{28}$$

This result is the same as in [26]. Substituting (19) into (28), we get

$$E_n = \frac{-\kappa_n \pm \sqrt{\kappa_n^2 - 4S_0^2\omega_n}}{2S_0^2}, \tag{29}$$

where

$$\kappa_n = -2\gamma\left(n + \frac{1}{2}\right)V_0\sqrt{S_0^2 - V_0^2} + 2mS_0V_0, \tag{30}$$

$$\omega_n = \gamma^2\left(n + \frac{1}{2}\right)(S_0^2 - V_0^2) - 2\gamma\left(n + \frac{1}{2}\right)mS_0\sqrt{S_0^2 - V_0^2} + m^2V_0^2. \tag{31}$$

Now we can calculate the corresponding unnormalized eigenfunctions by using the wave function generator given by (15)

$$f_n(s) = (-1)^n C_2 \frac{\Gamma(n + 2\varepsilon_n + 1)}{\Gamma(2\varepsilon_n + 1)} {}_1F_1(-n, 2\varepsilon_n + 1; -2\chi s), \tag{32}$$

where $\Gamma$ and ${}_1F_1$ are known as the gamma and the confluent hypergeometric functions, respectively [27]. Therefore, we can write the total radial wave function by using (24) and (32) as

$$u_n(s) = Ns^{\varepsilon_n}e^{\chi s}{}_1F_1(-n, 2\varepsilon_n + 1; -2\chi s), \tag{33}$$

where $N$ is normalization constant.

After we get the energy eigenvalues and eigenfunctions of the radial s-wave Klein-Gordon equation for the exponential scalar and vector potentials, we can consider some special cases. When the scalar potential stronger than the vector potential ($S_0 > V_0$), there is always a bound state. If we consider $S_0 \neq 0$ and $V_0 = 0$ case, the ground state energy is given by $E_0^2 = am - a^2/4$ for $m > a/4$. If the scalar potential is equal to the vector potential

($S_0 = V_0$), (16) reduces to a Schrödinger-type differential equation which solutions are represented by Bessel functions. If $S_0 < V_0$ and $S_0 = 0$, $V_0 \neq 0$, there is no bound state and the wave functions do not reach any limit since the parameter $K_1$ becomes a complex number.

## 4 Conclusion

In this article, we have shown that the radial s-wave Klein-Gordon equation can be solved exactly by using the asymptotic iteration method under exponential-type scalar and vector potentials where the scalar potential greater than its vector one. The wave functions are presented by confluent hypergeometric functions. The wave functions and energy eigenvalues are in good agreement with the result obtained by other methods. We have also discussed some special cases.

The asymptotic iteration method used in this paper is an elegant and powerful technique. If there are analytically solvable potentials, it provides the closed-forms for the energy eigenvalues as well as the corresponding eigenfunctions. But, if there is no such a solution, the energy eigenvalues are obtained by using an iterative approach [16, 19, 20]. It is worth extending this method to examine other potentials. The results are sufficiently accurate for practical purposes.

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